

There is no $(75, 32, 10, 16)$ strongly regular graph

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Abstract

We show that there is no $(75, 32, 10, 16)$ strongly regular graph. The result is obtained by a mix of algebraic and computational approaches. The main idea is to build large enough induced structure and apply the star complement technique. Our result implies that there is no regular two-graph on 76 vertices and no partial geometry with parameters $\text{pg}(4, 7, 2)$. In particular, it implies that there is no $(76, 35, 18, 14)$ strongly-regular graph. In order to solve this classification problem we also develop an efficient algorithm for the problem of finding a maximal clique in a graph.

1 Introduction

A popular notion in algebraic graph theory is the concept of a *strongly regular* graph. We say that a k -regular graph of order v and diameter 2 is strongly regular with parameters (v, k, λ, μ) if any pair of non-adjacent vertices has precisely λ common neighbors while two non-adjacent vertices share μ common neighbors.

A fundamental question about strongly regular graphs is for which parameters does a strongly regular graph exist? For example the notorious question about the existence of a graph of order 3250, girth 5, and diameter 2 (also known as Moore graph) asks for the existence of a strongly regular graph (from now on SRG) with parameters $(3250, 57, 0, 1)$. There are essentially four general ways to rule out certain parameters for SRG's from being realizable. An easy double-counting argument gives us the condition that

$$(v - k - 1)\mu = k(k - \lambda - 1).$$

The second condition requires that the numbers

$$\frac{1}{2} \left[(v-1) \pm \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right],$$

are integers, and comes from counting the multiplicities of the eigenvalues of a SRG.

Finally one way to rule out certain SRG is through the so-called Krein and absolute bounds see [8, pp. 231]. All the described criterion's still leave room for many parameters for which it is not known whether there exists such a SRG. The state of affairs for all possible parameters on up to 1300 vertices is tracked by Brouwer on his web site [5]. It can be seen that on up to 100 vertices there are essentially 15 parameters whose classification is still open, the smallest three being $(65, 32, 15, 16)$, $(69, 20, 7, 5)$, and $(75, 32, 10, 16)$. We found the parameters of the last one the most intriguing since the existence of the SRG with $(75, 32, 10, 16)$ is connected to the existence of certain so called two-graphs, and referred in [8, pp. 263] as one of the oldest open problems in this topic. Moreover, it is also connected to the existence of certain partial geometries.

Given that there is no general technique for deciding whether a certain parameter is realizable, a lot of effort has been put into establishing certain structural results about the missing SRG's. Specifically for a potential SRG X with parameters $(75, 32, 10, 16)$, Haemers and Tonchev [11] showed in 1996 that the chromatic number of X is at least 6. Four years later Makhnev showed [14] that X does not contain a 16-regular subgraph. Recently Behbahani and Lam [2] also derived some constraints about the structure of the automorphism group of X . Particularly, they showed that if p is a prime dividing $|\text{Aut}(X)|$, then $p = 2$ or $p = 3$.

In this paper we use the so called star-complement technique [6] in order to establish that in fact a SRG with parameters $(75, 32, 10, 16)$ does not exist. Moreover, since existence of such a graph is in one to one correspondence with the question of existence of regular two-graph on 76 vertices [8, pp. 249], our result implies that there is no regular two-graph with such order. The result has even further consequences: Define a partial geometry $\text{pg}(s, t, \alpha)$, for $s, t, \alpha \geq 1$, as incidence structure $C = (P, L, I)$ consistent of points P , lines L , and set $I \subset P \times L$ of incidences (we say that $p \in P$ is incident with $l \in L$ if $(p, l) \in I$), such that the following holds. Each pair of points has at most one line incident with both of them. Each line is incident with $s + 1$ points, while each point is incident with $t + 1$ lines. Finally, if a point p and a line l are not incident, there are exactly α pairs $(q, m) \in I$,

such that p is incident with m and q is incident with l . The point graph of $\text{pg}(s, t, \alpha)$ is the graph on its points with two points adjacent if they are incident with a common line. This graph is a strongly regular graph [4]. Up to now the existence of $\text{pg}(4, 7, 2)$ was unknown, but it was known that the point graph of this geometry must be a SGR with parameters $(75, 32, 10, 16)$. Therefore our result shows that also $\text{pg}(4, 7, 2)$ does not exist. As an application of Seidel switching one can also deduce that the non-existence of a $(75, 32, 10, 16)$ SRG implies that there is no $(76, 35, 18, 14)$ SRG [9].

The star-complement technique turned out to be a useful tool for re-proving classification results for some SRG's [16, 17] although its direct application fails for SRG's with a large number of vertices or valency. The two main drawbacks being the large search space for induced subgraphs and the problem of computing the clique number of some large graphs.

In this paper we address both problems by presenting an application of the interlacing principle that is very effective at pruning the underlying search space as well as an efficient algorithm for finding the clique number. The algorithm for computing the clique number is crucial since computing the clique number of most of our graphs is computationally infeasible for all state of the art programs.

The paper is organized as follows. In the next section we introduce the star-complement technique and the interlacing criterion that we use. In the following two sections we use the star complement technique in order to first establish the clique number of X and finally show that X does not exist. We finish by presenting the obtained results and discussing some of the computational aspects.

2 Preliminaries

2.1 Star complements

The idea behind star-complements revolves around the notion of a so called *star-complement graph*. Let G be a simple graph of order n , A_G its adjacency matrix and r one of its eigenvalues with multiplicity f . We will say that r is an eigenvalue of G whenever we mean that r is an eigenvalue of A_G .

An induced subgraph $H \subseteq G$ is called a *star-complement for G and eigenvalue r* if it has order $n - f$ and r is not an eigenvalue of H . As it turns out [6], there is a star-complement for every eigenvalue of G . For convenience we record this fact in the following proposition.

Proposition 1. *If G is a graph and r an eigenvalue of G , then G has a*

star complement for r .

Before explaining the role of star-complements, let us mention that one can construct a star-complement for an eigenvalue r by extending an induced subgraph of G that does not contain r as an eigenvalue [16, Lemma 3]. More precisely:

Proposition 2. *Let G be a graph with eigenvalue r . If H' is an induced subgraph of G that does not contain r as an eigenvalue then there exist a star complement H for G and eigenvalue r so that H' is an induced subgraph of H .*

The main motivation of star-complements is that they in some way allow us to reconstruct G . The reader can find the precise implications in [6, pp. 150], in this paper we shall formulate the theory to suit the needs of our application.

Let H be a star-complement for G with eigenvalue r and define the inner product

$$\langle u, v \rangle = u(rI - A_H)^{-1}v^t.$$

The *comparability graph* of H and r denoted by $\text{Comp}(H, r)$ is the graph with vertex set

$$V(\text{Comp}(H, r)) = \{u \in \{0, 1\}^{n-f} \mid \langle u, u \rangle = r \text{ and } \langle u, \vec{1} \rangle = -1\},$$

and adjacency defined as

$$u \sim v \iff \langle u, v \rangle \in \{-1, 0\}.$$

Let us remark that the condition that the inner product of a vertex of $\text{Comp}(H, r)$ and the all-ones vector is -1 does not hold in general but only if we assume that G is a regular graph, see [19].

As it turns out, the problem of constructing G is reduced to the problem of finding cliques in $\text{Comp}(H, r)$. Specifically

Proposition 3. *If r is an eigenvalue of G with multiplicity f , H a star complement for G and r , then $\text{Comp}(H, r)$ has a f -clique.*

This already sets the general idea behind the application of the star complement technique. Suppose G is a SRG with parameters (v, k, λ, μ) and r an eigenvalue of G with (large) multiplicity f . Suppose that we know that H' is a induced subgraph of G and does not have r as an eigenvalue. If H' is large enough we can compute its comparability graph and check for

f -cliques. If the obtained graph does not have such a clique then G does not exist.

In most cases we cannot directly find an induced subgraph H' large enough to be a star complement. In that case, by Proposition 2, we can extend H' in all possible ways to obtain candidates for a star-complement of G and r . Depending on how large H' is, we may obtain a large set of candidates, and for each such candidate H we need to compute the respective clique number of $\text{Comp}(H, r)$.

The set of all possible candidates for star-complements gets large very quickly and hence we need an efficient pruning method that we explain in the next subsection.

2.2 Interlacing

If $n > m$ and $\lambda_1 \geq \dots \geq \lambda_n$, $\mu_1 \geq \dots \geq \mu_m$ are two sequences of real numbers, we say that $\{\mu_i\}_{i=1}^m$ *interlaces* $\{\lambda_i\}_{i=1}^n$ if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for } i \in \{1, \dots, m\}.$$

The well-known interlacing principle states that the eigenvalues of an induced subgraph of G interlace the eigenvalues of G . As it turns out this is not a very effective condition for pruning the induced subgraphs obtained in our application. A much more effective criterion is to use the ‘partitioned’ version of the interlacing principle [10, Cor. 2.3] which we state as follows. Suppose $\mathcal{V} = (\mathcal{V}_1, \dots, \mathcal{V}_k)$ is a partition of the vertices of G . Let $e(\mathcal{V}_i, \mathcal{V}_j)$ denote the number of edges between the vertices of \mathcal{V}_i and \mathcal{V}_j if $i \neq j$, and the number of edges in the graph induced by \mathcal{V}_i otherwise. Consider the $k \times k$ matrix $A_{\mathcal{V}} = (a_{i,j})_{i,j=1}^k$ where

$$a_{i,j} = \begin{cases} \frac{e(\mathcal{V}_i, \mathcal{V}_j)}{|\mathcal{V}_i|} & \text{if } i \neq j \\ \frac{2e(\mathcal{V}_i)}{|\mathcal{V}_i|} & \text{if } i = j \end{cases}.$$

As it turns out, the eigenvalues of $A_{\mathcal{V}}$ interlace the eigenvalues of G .

Proposition 4. *Let G be a graph and \mathcal{V} a partition of its vertices. Then, the eigenvalues of G are interlaced by the eigenvalues of $A_{\mathcal{V}}$.*

Given that we want to decide whether a graph H with vertex set $\{v_1, \dots, v_k\}$ is an induced subgraph of G we consider the partition

$$\mathcal{V} = (\{v_1\}, \dots, \{v_k\}, V(G) \setminus V(H)).$$

Since in our scenario G is regular, we can always compute the number of edges in $V(G) \setminus V(H)$ as well as between the vertices of $V(H)$ and $V(G) \setminus V(H)$. We call the matrix that is obtained from H by using this partition *the partitioned adjacency matrix* of H .

From Proposition 4 we have that the eigenvalues of the partitioned matrix of H must interlace the eigenvalues of G . This turns out to be quite a sharp pruning condition which we will call the *interlacing condition*. In particular if the interlacing condition of a graph H is not satisfied, we shall say that H does not interlace G .

Let us remark that this condition appears to be more efficient than the positive-definiteness criterion used in [7]. More precisely, we were able to find graphs such that the pruning condition used in [7] is satisfied but they do not interlace our graph. We were however not able to find graphs satisfying the converse situation. However, the drawback of this criterion is that the matrices in question are not symmetric and hence computing their eigenvalues is a much less efficient task.

2.3 Approach

The problem of determining whether a (v, k, λ, μ) SRG graph G exists is thus reduced to the following. Pick an eigenvalue r of G with large multiplicity. Start with a large induced subgraph H' that does not have r as an eigenvalue and must appear as an induced subgraph of G . Extend H' to a star-complement of G and r using the described pruning conditions to get rid of invalid graphs. Finally, for all potential star-complements H compute the clique number of $\text{Comp}(H, r)$. In practice, $\text{Comp}(H, r)$ can be a very large and dense graph and we explain how to compute its clique number in Section 5.

Now, let us describe our approach for the classification of SRG with parameters $(75, 32, 10, 16)$. For the eigenvalue we take $r = 2$ and look to find a small list \mathcal{L} with graphs of large order such that at least one member of \mathcal{L} is an induced subgraph of a $(75, 32, 10, 16)$ SRG X . When the list is obtained, we proceed to show that no graph in \mathcal{L} is an induced subgraph of X as follows. For $H \in \mathcal{L}$ let $\text{sc}(H)$ be a largest induced subgraph of H that does not have 2 as an eigenvalue and has order at most 19. Note that $\text{sc}(H)$ may not be unique and in this case we can pick an arbitrary such subgraph. If $|V(\text{sc}(H))| = 19$ then $\text{sc}(H)$ is a star complement for X and we use the theory described above to verify that $\omega(\text{Comp}(\text{sc}(H), 2)) < 56$, and hence that H is not an induced subgraph of X . If $|V(\text{sc}(H))| < 19$ then we extend $\text{sc}(H)$ by adding $19 - |V(\text{sc}(H))|$ vertices in all possible ways as to obtain

(by Proposition 2) a list of possible star complements for X . Again, we show that none of the obtained star complements has a comparability graph with a large enough clique. The process of extending an induced subgraph H to a graph of order 19 is done by inductively introducing new vertices in all possible ways, and in the end removing all candidates that have 2 as eigenvalue or do not interlace. In order to minimize the list of candidate graphs we also make use of the following observation. Suppose that there is a pair of vertices $u, v \in V(H)$ that does not yet have many common neighbors in the induced subgraph - that is $u \sim v$ and $|N(u) \cap N(v)| < \lambda = 10$, or $u \not\sim v$ and $|N(u) \cap N(v)| < \mu = 16$. Suppose further that for every $S \subset V(H) \setminus \{u, v\}$ all the graphs obtained by adding a new vertex adjacent to $S \cup \{u, v\}$ that interlace X also do not contain 2 as an eigenvalue. Let us say that such a pair of vertices is called *graceful*.

In virtue of Proposition 2 we can simply use these graphs when building a complete list of star complements of X having H as subgraph. Stating it as a proposition

Proposition 5. *If u, v is a graceful pair for H and \mathcal{L} a list of all graphs obtained by adding a new vertex x to H that is joined to u, v and a subset of $V(H) \setminus \{u, v\}$. Then there exist a star complement G for X such that at least one of the members of \mathcal{L} is an induced subgraph of G .*

The described approach is performed by the program *extend.c* that we describe later. In particular, it turns out that the above procedure is computationally feasible if the list \mathcal{L} of induced subgraphs does not include graphs that are, when reduced to a subgraph without eigenvalue 2, of order less than 17. In practice, this is almost the same as demanding that for each $G \in \mathcal{L}$ we have $n(G) - k_2(G) \geq 17$, where $n(G)$ is order of G and $k_2(G)$ is the multiplicity of eigenvalue 2 in G .

3 Cliques in a SRG with parameters $(75, 32, 10, 16)$

In what follows let X denote a possible strongly regular graph with parameters $(75, 32, 10, 16)$. Our main goal is to prove that X does not exist. In order to do so we first establish a structural claim related to its cliques. Notice that Hoffman bound [8, pp. 204] implies that \overline{X} has independence number at most 5 and hence that X has clique number at most 5. On the other hand, Bondarenko, Prymak, and Radchenko developed a general tool for bounding the number of 4-cliques in a strongly regular graph [3]. In particular, they have established that a SRG with parameters $(75, 32, 10, 16)$ has at least 783 4-cliques.

In this section we show that in fact X has clique number 5, more precisely, we show the following result.

Proposition 6. *If X exists, its clique number is 5. Moreover, every 4-clique of X is contained in a 5-clique.*

In order to prove the result we need to recall a very useful lemma whose proof the reader may find in [3]. Let H be an induced subgraph of order m of a (v, k, λ, μ) strongly regular graph G , and let $(d_0, d_1, \dots, d_{m-1})$ be a vector such that d_i denotes the number of vertices of H having degree i . Similarly let (b_0, \dots, b_m) be a vector where b_i denotes the number of vertices of $G - H$ that have i neighbors in H . The next lemma gives a relationship between these numbers.

Lemma 1. *With notation as above, the following three equations hold*

$$\sum_{i=0}^m b_i = v - m,$$

$$\sum_{i=0}^m i b_i = m k - \sum_{j=0}^{m-1} j d_j,$$

and

$$\sum_{i=0}^m \binom{i}{2} b_i = \binom{m}{2} \mu - \sum_{i=0}^{m-1} \binom{i}{2} d_i + \frac{1}{2}(\lambda - \mu) \sum_{i=0}^{m-1} i d_i.$$

Suppose now that X has a 4-clique K_4 that is not contained in a 5-clique. Letting $H = K_4$ and applying the above Lemma it can easily be verified that there are 4 solutions (b_0, b_1, b_2, b_3) (notice that by our assumption $b_4 = 0$) to the above system, namely

$$(3, 20, 48, 0), (0, 29, 39, 3), (1, 26, 42, 2) \text{ and } (2, 23, 45, 1). \quad (1)$$

In what follows we analyze all these possibilities, showing that none of these solutions occurs as a configuration in X . We split the proof into four sections each dealing with a different value of (b_0, b_1, b_2, b_3) . The general idea is to use the structure given by a specific configuration to find a small list of graphs that must be an induced subgraph of X . For each possible case we have written simple Sage [20] programs that build graphs with the established structure and prune them using the interlacing principle we described. Whenever we assert that some induced structure is not possible or say that there is a list of graphs satisfying it, there is a corresponding Sage

program that computed this part of the claim. Each such program (with the respective output) is recorded in Table 1. Throughout the rest of the paper K_4 will denote a 4-clique of X .

In the next section we shall analyze the case when X has clique number 5. Since we will also need a glimpse into this case already in this section, let us remark at this point that if K_5 is a 5-clique of X then every vertex in $V(X) - V(K_5)$ has precisely two neighbors in K_5 . One way to see this claim is to use the above system of equities which only gives $(0, 0, 70, 0, 0, 0)$ as a solution vector. Finally, let us remark that throughout the paper we will use the notation $X[S]$ to denote the subgraph of X induced by the set of vertices $S \subseteq V(X)$.

3.1 Case $(3, 20, 48, 0)$

Let us denote with X_0, X_1, X_2 the subsets of vertices in $V(X) \setminus V(K_4)$ that have, respectively, 0, 1, and 2 neighbors in K_4 . Moreover, denote the vertices in X_0 by x_1, x_2, x_3 .

Lemma 2. *Every vertex in X_2 has precisely two neighbors in X_0 .*

Proof. Let $x_i \in X_0$. We use an argument that will be repeatedly used in this paper. Since x_i is not adjacent to any of the vertices in K_4 it has to have 16 common neighbors (since X is strongly regular with $\mu = 16$) with each of its vertices. Thus there are $4 \cdot 16$ paths of length 2 from x_i to K_4 . On the other hand, x_i has 32 neighbors (X is 32-regular) in $X_0 \cup X_1 \cup X_2$. All the neighbors are in fact in X_2 , for otherwise they could not form 64 2-paths to K_4 . Since $|N(x_i) \cap N(x_j)| = 16$ and $|N(x_i)| = 32$, we have

$$48 \geq |N(x_1) \cup N(x_2) \cup N(x_3)| = 3 \cdot 32 - 3 \cdot 16 + |N(x_1) \cap N(x_2) \cap N(x_3)|,$$

by the inclusion-exclusion principle. Thus $N(x_1) \cap N(x_2) \cap N(x_3) = \emptyset$. Therefore every vertex in X_2 is adjacent to precisely two vertices in X_0 . \square

For $i \in \{1, 2, 3\}$ let $X_2^{i,j} \subseteq X_2$ be the graphs induced by $N(x_i) \cap N(x_j)$.

Lemma 3. *The graphs $X_{1,2}, X_{1,3}, X_{2,3}$ have order 16 and are isomorphic to the disjoint union of cycles.*

Proof. It is clear that the sets have order 16. Let $v \in X_{1,2}$. We count the number of 2-paths from v to K_4 . Since v is adjacent to 2 vertices of K_4 , there must be $2 \cdot 10 + 2 \cdot 16$ such paths. Denote with k the number of neighbors of v in X_2 . By the previous lemma, v is adjacent to 2 vertices in X_0 , thus

it is adjacent to $32 - 2 - 2 - k$ vertices in X_1 . By counting 2-paths we thus have:

$$2 \cdot 10 + 2 \cdot 16 = 2k + 1(28 - k) + 0 \cdot 2 + 2 \cdot 3.$$

Therefore, v has 18 neighbors in X_2 , and since it is not adjacent to x_3 it must have 16 neighbors in $X_2 - X^{1,2} = N(x_3)$. This implies that v has precisely 2 neighbors in $X_{1,2}$. \square

In [14] it was shown that if X exists, then it does not have a 16-regular subgraph. By removing the disjoint union of cycles from the graph $X[X_2]$ we obtain a 16-regular subgraph and hence this configuration is impossible.

3.2 Case (1, 26, 42, 2)

Let X_0, X_1, X_2, X_3 be the sets of vertices having 0, 1, 2, and 3 neighbors in K_4 , respectively. In particular, let $x_0 \in X_0$ and $x_1 \neq x_2 \in X_3$. In what follows we prove a series of claim describing the structure of a graph with this configuration.

Lemma 4. *Vertices x_1 and x_2 are not adjacent.*

Proof. Suppose $x_1 \sim x_2$. There are up to isomorphism only two possible induced graphs on $K_4 \cup \{x_1, x_2\}$. Moreover, if we add the vertex x_0 we obtain 6 candidate graphs for an induced subgraph of X . None of them interlaces X , which was an easy task to check by computer. \square

Lemma 5. *Vertex x_0 is adjacent to both vertices in X_3 . Moreover, it has 2 neighbors in X_1 and 28 neighbors in X_2 .*

Proof. For the sake of contradiction, suppose x_0 is adjacent to $k \in \{0, 1\}$ vertices of X_3 . Let t be the number of neighbors of x_0 in X_1 . By double counting 2-paths from x_0 to K_4 we obtain:

$$4 \cdot 16 = 3k + t + 2(32 - k - t),$$

which gives that $k = t$. Without loss of generality suppose that x_1 is not adjacent to x_0 . By counting the number of 2-paths in a similar way we obtain that x_1 has 8 neighbors in X_2 . But by strong regularity, x_0 and x_1 must have 16 common neighbors which is not possible since x_0, x_1 can share at most $k \leq 1$ common neighbors in X_1 and 8 common neighbors in X_2 . Hence x_0 is adjacent to both x_1 and x_2 and so $k = t = 2$ and the claim follows. \square

In virtue of Lemma 5, let x'_0, x''_0 be the vertices in X_1 that are adjacent to x_0 .

Lemma 6. *Vertices x_1, x_2 each have 19 neighbors in X_1 and 9 neighbors in X_2 . In particular, 12 or 13 vertices of X_1 are adjacent to both x_1 and x_2 , 6 or 7 vertices only to x_1 , and 6 or 7 only to x_2 .*

Proof. The first part of the claim is an easy application of the already used double counting argument. Let now N_{x_1} and N_{x_2} be the neighbors of x_1 and x_2 in X_1 respectively. Since $|N_{x_1} \cup N_{x_2}| \leq 26$ (the size of X_1), we must have $|N_{x_1} \cap N_{x_2}| \geq 12$ by the inclusion-exclusion principle. On the other hand, x_1 and x_2 must have 16 common neighbors. Since x_0 is a common neighbor and they have 2 or 3 common neighbors on K_4 it follows that $|N_{x_1} \cap N_{x_2}| \leq 13$. The other assertions follow easily. \square

Lemma 7. *For $i = 1, 2$, the vertex x_i is adjacent to at least one of the vertices in $\{x'_0, x''_0\}$.*

Proof. For $i = 1, 2$, the vertex x_i has 10 common neighbors with x_0 . By Lemma 6, x_i only has 9 vertices in X_2 , thus it must be adjacent to at least one of x'_0, x''_0 . \square

Let X_2^{-0} be the set of vertices in X_2 that are not adjacent to x_0 . By Lemma 5, $|X_2^{-0}| = 14$.

Lemma 8. *At most one vertex from X_2^{-0} is adjacent to x_1 , and at most one is adjacent to x_2 .*

Proof. Vertex x_1 shares at most 2 common neighbors with x_0 in X_1 (possibly x'_0 or x''_0). Thus it must have at least 8 out of 9 neighbors in X_2 adjacent to x_0 . By symmetry, the claim holds for x_2 . \square

Lemma 9. *Each vertex in $X[X_2^{-0}]$ that is not adjacent to any of the vertices in $\{x_1, x_2\}$ has degree $t \leq 2$ and it has t neighbors in $\{x'_0, x''_0\}$. Vertices (at most two) in X_2^{-0} that are adjacent to exactly one of x_1 and x_2 have degree $t - 1$ in $X[X_2^{-0}]$ and $t \geq 1$ neighbors in $\{x'_0, x''_0\}$. If there exists a vertex in X_2^{-0} that is adjacent to both in x_1 and x_2 , then it is adjacent to both vertices in $\{x'_0, x''_0\}$ and has degree 0 in $X[X_2^{-0}]$.*

Proof. Let $v \in X_2^{-0}$. Notice that v must have 16 common neighbors with x_0 . First, assume it is not adjacent to x_1 or x_2 . Then their common neighbors can only be in $\{x'_0, x''_0\}$, say t of them, and in $X_2 \setminus X_2^{-0}$. By double counting

2-paths from v to K_4 we obtain that v has 16 neighbors in X_2 . Thus t of them must be in X_2^{-0} .

Second, assume that v is adjacent to exactly one of the x_1, x_2 (in this case it can be adjacent to only one of them). Then it has $16 - 1 - t$ neighbors in $X_2 \setminus X_2^{-0}$. On the other hand, by double counting, its degree in $X[X_2]$ is 14. Thus its degree in $X[X_2^{-0}]$ is $t - 1$.

Finally, if v is adjacent to x_1 and x_2 , it has degree 12 in $X[X_2]$, thus all this neighbors have to be in $X_2 \setminus X_2^{-0}$ and it also has to be adjacent to both vertices in $\{x'_0, x''_0\}$. \square

Lemma 10. *Each of the vertices x'_0, x''_0 has $15 - t$ neighbors in X_2^{-0} , where $t \in \{1, 2\}$ is the number of its neighbors in $\{x_1, x_2\}$. Moreover, x'_0 and x''_0 are not adjacent.*

Proof. By double counting 2-paths from x'_0 to K_4 we have that x_0 has $25 - 2t$ neighbors in X_2 . Vertices x_0 and x'_0 have 10 common neighbors. Let s be equal to 1 if x'_0 and x''_0 are adjacent and 0 otherwise. x_0 and x'_0 must have $10 - t - s$ common neighbors in X_2 , thus x'_0 has $24 - 2t - (10 - t - s) = 15 - t + s$ neighbors in X_2^{-0} . Similar holds for x''_0 and since $|X_2^{-0}| = 14$, x'_0 and x''_0 have more than 10 common neighbors. Thus they are not adjacent and $s = 0$. The lemma holds. \square

The above lemmas give enough structure to be able to computationally obtain a small list of graphs that interlace X and must be induced subgraph of X , provided that X contains a K_4 with this configuration.

Proposition 7. *There are 3597 graphs of the form $K_4 \cup \{x_0, x'_0, x''_0, x_1, x_2\} \cup X_2^{-0}$ that interlace X .*

3.3 Case (2, 23, 45, 1)

Let $X_0 = \{x_0, x_1\}$, $X_1, X_2, X_3 = \{x_3\}$ be the sets of vertices having 0, 1, 2, and 3 neighbors in K_4 respectively. Again, we start by proving certain structural claims about this configuration.

Lemma 11. $x_0 \not\sim x_1$.

Proof. If $x_0 \sim x_1$ then the number of 2-paths from x_0 to K_4 is at most $3 + 2 \cdot 30$. But since x_0 is not adjacent to any vertex of K_4 , it should have precisely $4 \cdot 16$ 2-paths to it. \square

Lemma 12. $x_0 \sim x_3$ and $x_1 \sim x_3$.

Proof. Suppose x_0 is not adjacent to x_3 and let N_0, N_1 , respectively, be the sets of neighbors of x_0, x_1 in X_2 . By double counting 2-paths to K_4 from x_0 or x_1 we have $|N_0| = 32$ while $|N_1| = 32 - 2t$ where $t \in \{0, 1\}$ depending on whether x_1 is adjacent to x_3 or not. Since all the neighbors of x_0 are in X_2 , we have $|N_0 \cap N_1| = 16$. But this implies $|N_0 \cup N_1| = 48 - 2t \geq 46$ which is not possible as X_2 has size 45. \square

Lemma 13. *Vertices x_0 and x_1 have precisely one neighbor in X_1 . In particular, the two neighbors are distinct.*

Proof. Let t be the number of neighbors of x_0 in X_1 . By counting 2-paths from x_0 to K_4 we have

$$16 \cdot 4 = 3 + t + 2 \cdot (32 - 1 - t),$$

which gives that $t = 1$, and x_0 has 30 neighbors in X_2 . Same holds for x_1 . Let again N_0, N_1 be the sets of neighbors of x_0, x_1 in X_2 . Then $|N_0 \cup N_1| \leq 45$, thus $|N_0 \cap N_1| \geq 15$. Since x_0 and x_1 have a common neighbor x_3 , $|N_0 \cap N_1| = 15$ and $|N_0 \cup N_1| = 45$. This implies also that x_0 and x_1 cannot have a common neighbor in X_1 . \square

The last two lemmas now imply that X_2 can be partitioned into sets $X_2^0, X_2^{\{0,1\}}, X_2^1$ each of size 15 such that every vertex in X_2^i is adjacent to x_i and not adjacent to x_{1-i} for $i = 0, 1$ and every vertex in $X_2^{\{0,1\}}$ is adjacent to both x_0 and x_1 . Let us denote the neighbors of x_0, x_1 in X_1 by x'_0 and x'_1 respectively.

Lemma 14. *If x_3 is adjacent to x'_1 then it has 1 neighbor in X_2^0 , otherwise it has no neighbor in X_2^0 .*

Proof. x_3 has 10 neighbors in X_2 by double counting of 2-paths to K_4 . On the other hand it must have 10 common neighbors with x_1 . Notice that the common neighbors can only be in $X_2 \cup \{x'_1\}$. If x_3 is adjacent to x'_1 , then it must have 9 neighbors in $X_2^{0,1} \cup X_2^1$ thus 1 in X_2^0 . On the other hand, if x_3 is not adjacent to x'_1 , then it must have all 10 neighbors in $X_2^{0,1} \cup X_2^1$ and no neighbor in X_2^0 . \square

Lemma 15. *The graph $X[X_2^0]$ has maximal degree 2. If $v \in X_2^0$ has degree 2, then it is not adjacent to x_3 but it is with x'_1 , if it has degree 1, it is adjacent with either both x_3 and x'_1 or none of them, and if it has degree 0, it is adjacent to x_3 and not adjacent with x'_1 .*

Proof. Pick a vertex $v \in X_2^0$ and let $s \in \{0, 1\}$ indicate if it is adjacent with x'_1 and $t \in \{0, 1\}$ indicate if it is adjacent with x_3 . Let r be the number of neighbors of v in X_2 . We count 2-paths from v to K_4 :

$$2 \cdot 10 + 2 \cdot 16 = 2 \cdot 3 + 3t + 2r + (32 - 2 - t - r - 1).$$

Thus $r = 17 - 2t$. Vertices v and x_1 must have 16 common neighbors. This implies that the number of neighbors of v in $X_2^{0,1} \cup X_2^1$ is $16 - t - s$. Hence we have that v has $(17 - 2t) - (16 - t - s) = 1 - t + s$ neighbors in X_2^0 and hence the lemma follows. \square

By generating graphs with the established structure we infer that none of them interlaces X .

Proposition 8. *No graph of the form $X_2^0 \cup \{x_0, x_1, x'_1, x_3\} \cup K_4$ interlaces X .*

3.4 Case (0, 29, 39, 3)

Let X_1, X_2 and $X_3 = \{x_0, x_1, x_2\}$ be the respective subsets of vertices of X . There are three non-isomorphic ways to introduce 3 vertices to K_4 by joining each vertex to 3 vertex of K_4 . Each such graph G_1, G_2, G_3 can be uniquely described by a tuple $\vec{n} = (n_1, n_2, n_3, n_4)$ counting the number of edges from the i 'th vertex of K_4 to X_3 . By relabeling the vertices of K_4 if needed we obtain three tuples (0, 3, 3, 3), (1, 3, 3, 2) and (2, 2, 2, 3). We proceed by establishing certain structural claims about this configuration.

Lemma 16. *The vertices of X_3 form an independent set.*

Proof. No matter how we introduce edges among the vertices of X_3 in the graph $K_4 \cup X_3$ we do not obtain a graph interlacing X . \square

Lemma 17. *Every vertex $x \in \{x_0, x_1, x_2\}$ has 21 neighbors in X_1 and 8 neighbors in X_2 .*

Proof. Let x have k neighbors in X_1 and $l = 32 - 3 - k$ neighbors in X_2 . We count the number of paths of length 2 from x to K_4 . We have $3 \cdot 10 + 16 = 3 \cdot 3 + k + 2 \cdot (32 - k - 3)$ and thus $k = 21$ and $l = 8$. \square

Let X_2^0, X_2^1, X_2^3 be the neighbors in X_2 of x_0, x_1, x_2 respectively. We have proved that $|X_2^0| = |X_2^1| = |X_2^3| = 8$. Notice that the sets X_2^0, X_2^1, X_2^3 need not be disjoint.

Lemma 18. *It holds $|X_2^0 \cap X_2^1|, |X_2^0 \cap X_2^3|, |X_2^1 \cap X_2^3| \in \{0, 1\}$.*

Proof. Vertices x_0 and x_1 have 16 common neighbors, at least 2 of them are on K_4 . Each of x_0, x_1 has 21 neighbors in X_1 , where $|X_1| = 29$. Thus they must have at least 13 common neighbors in X_1 . The latter implies that they have at most one common neighbor in X_2 . Same holds for all other pairs from the assertion of the lemma. \square

Lemma 19. *For $i \in \{1, 2\}$ the graph $X[X_2^0 \setminus X_2^i]$ is triangle-free.*

Proof. Assume that there exists a triangle in $X[X_2^0 \setminus X_2^i]$. Together with x_0 it forms a K_4 . Assume this K_4 does not extend to a K_5 . Vertex x_i is not adjacent to any of the vertices of this K_4 . Thus we have have a 4-clique with some vertices that are not adjacent to it. We have already covered this configuration before and shown that it is not feasible. On the other hand, if K_4 extends to a K_5 , we have a K_5 and a vertex that is adjacent to at most one vertex on it. Again, this is not possible, since every vertex not on K_5 must have precisely two neighbors on K_5 . \square

Let X_1^{-0} denote the subgraph of X_1 induced on all the vertices not adjacent to x_0 and let X_1^i be the set of vertices in X_1 adjacent to x_i for $i \in \{0, 1, 2\}$.

Lemma 20. *The graph X_1^{-0} has 8 vertices. At most one of the vertices in X_1^{-0} is not adjacent to x_1 and at most one is not adjacent to x_2 . Moreover, the graph on $X_1^{-0} \setminus X_1^i$, for $i \in \{1, 2\}$, has no triangles. Each vertex $v \in X_1^{-0}$ has degree k in X_1^{-0} at most 3 and is adjacent to precisely $9 - t - m + k$ vertices in X_2^0 , where $t \in \{0, 1, 2\}$ is the number of vertices adjacent to v in $\{x_1, x_2\}$ and $m \in \{0, 1\}$ the number of common neighbors of v and x_0 on K_4 .*

Proof. By Lemma 17, $|X_1^0| = 21$, thus $|X_1^{-0}| = 8$. For $i \in \{1, 2\}$, x_i and x_0 share at least 2 neighbors on K_4 . Since they are non-adjacent, they share 16 neighbors, thus at most 14 in X_1 . This implies $|X_1^0 \cap X_1^i| \leq 14$, thus $|X_1^0 \cup X_1^i| \geq 21 + 21 - 14 = 28$. Since $|X_1| = 29$ we see that there exist at most one vertex in X_1 that is not adjacent to x_0 and to x_i .

If there is a triangle in the graph induced by $X_1^{-0} \setminus X_1^i$, this triangle forms a K_4 with x_0 , while x_i is not adjacent to any of its vertices. If this 4-clique is not a part of a 5-clique the assertion follows since this case has already been dealt with (a K_4 with some vertices not adjacent to it). On the other hand, if this K_4 is a part of a K_5 , we have an induced subgraph of K_5 together with a vertex that is adjacent to one or none of the vertices on K_5 . Again, this is not possible since every vertex not on K_5 has precisely two neighbors in $V(K_5)$. Hence $X_1^{0,2}$ is indeed triangle-free.

Let now $v \in X_1^{-0}$ be as in the lemma. Denote with j the number of its neighbors in X_1 . By counting 2-paths to K_4 we get

$$10 + 3 \cdot 16 = 3 + 3t + j + 2(32 - 1 - t - j),$$

hence $j = 7 + t$. Denote with l the number of neighbors of v in X_2^0 . Vertex v and x_0 have 16 common neighbors, thus

$$16 = (j - k) + l + m = (7 + t - k) + l + m,$$

from which we get $9 - t - m + k = l \leq 8$. This implies also that $k \leq 2$. \square

Lemma 21. *Let v be a vertex of $X[X_2^0]$ with degree k , $t \in \{0, 1, 2\}$ neighbors in $\{x_1, x_2\}$, and $m \in \{1, 2\}$ the number of common neighbors of v and x_0 on K_4 . Then:*

$$k + m + t \leq 3.$$

In particular $k \leq 2$.

Proof. Let $v \in X_2^0$. Denote with j the number of neighbors of v in X_1 . By counting 2-paths from v to K_4 we get:

$$2 \cdot 10 + 2 \cdot 16 = 2 \cdot 3 + 1 \cdot 3 + 3t + j + 2(32 - 2 - 1 - t - j),$$

thus $j = 15 + t$. Let now $l \leq 8$ be the number of neighbors of v in X_1^{-0} . Vertices v and x_0 have 10 common neighbors:

$$10 = k + m + (j - l) = k + m + (15 + t - l),$$

thus $5 + k + m + t = l \leq 8$ and $k + m + t \leq 3$. Since $m \in \{1, 2\}$, $k \leq 2$. \square

By generating all graphs induced by $K_4 \cup \{x_0, x_1, x_2\} \cup X_2^0 \cup X_1^{-0}$ we obtain

Proposition 9. *There are 18089 non-isomorphic graphs of the form $K_4 \cup \{x_0, x_1, x_2\} \cup X_2^0 \cup X_1^{-0}$ that interlace X .*

The case analysis carried in this section resulted in a list of 21686 graphs, resulting in 6688644 star complements and roughly 40000 comparability graphs. By verifying that they all have clique number smaller than 56 we established Proposition 6.

Claim	Program	Output
Lemma 4	K4/126422/Claim-1.sage	/
Proposition 7	K4/126422/Case126422.sage	K4/12622/cands126422.g6
Proposition 8	K4/223451/Case223451.sage	/
Lemma 16	K4/029393/Claim1.sage	/
Proposition 9	K4/029393/generateFinal.sage	K4/029393/cands029393.g6
Lemma 24	K5/triangles/extendTriangle.sage	K5/triangles/candsTriag.g6

Table 1: Sage programs constructing small induced structure.

4 Main result

Let K_5 be a 5-clique of X with vertex set $\{k_1, \dots, k_5\}$. The only possible configuration of vertices not in K_5 is $(0, 0, 70, 0, 0, 0)$, therefore every vertex of X that is not in K_5 has precisely two neighbors in K_5 . For $1 \leq i < j \leq 5$, let $X_{i,j}$ be vertices in $V(X) \setminus V(K_5)$ that are adjacent to k_i and k_j . Since k_i and k_j are adjacent, they must have 10 common neighbors, 3 of them already on K_5 . Hence $V(G) \setminus V(K_5)$ is partitioned into 10 sets of 7 vertices, namely $X_{0,1}, X_{0,2}, \dots, X_{4,5}$. In what follows we establish structural results about these partitions.

Lemma 22. *For any $1 \leq i < j \leq 5$ the graph $X_{i,j}$ is either $\overline{K_7}$ or $K_3 \cup \overline{K_4}$ or $K_1 \cup K_3 \cup K_3$.*

Proof. Assume there exists an edge $e = \{x, y\}$ in the graph $X_{i,j}$. Then the vertices $\{x, y, k_i, k_j\}$ induce a 4-clique. By the result of the previous section, every 4-clique is contained in a 5-clique. Clearly, the additional vertex must be in $X_{i,j}$. Hence we have proved that every edge e in $X_{i,j}$ is contained in a triangle in $X_{i,j}$. Let T be a triangle in $X_{i,j}$ and $v \in X_{i,j}$ a vertex not on T . Since $T \cup \{k_i, k_j\}$ induces a 5-clique, every vertex not on this 5-clique is adjacent to exactly 2 vertices on this clique. Since v is adjacent to k_i and k_j , it is not adjacent to T and the lemma follows. \square

As it turns out every pair of triangles in distinct partitions $X_{i,j}, X_{k,l}$ induce quite a regular structure.

Lemma 23. *Let $1 \leq i < j \leq 5$, $1 \leq k < l \leq 5$ and let T, T' be two triangles of $X_{i,j}$ and $X_{k,l}$, respectively. Let $c = |\{i, j, k, l\}|$. If $c = 3$, then the edges from T to T' form a perfect matching. If $c = 4$, they form a complement of a perfect matching.*

Proof. First assume $c = 3$. Since $T \cup \{k_i, k_j\}$ forms a 5-clique, every vertex of T' is adjacent to exactly 2 vertices in this 5-clique. Since $c = 3$, it must be adjacent to exactly one vertex in T . Similarly, every vertex of T must be adjacent to exactly one vertex in T' . Thus, the edges from T to T' form a perfect matching. The case when $c = 4$ is similar. \square

Our next lemma shows that not all partitions $X_{i,j}$ contain a triangle. In fact at most 7 do.

Lemma 24. *There are at least three distinct pairs $\{i, j\}, \{k, l\}, \{m, n\}$ such that $X_{i,j}, X_{k,l}$ and $X_{m,n}$ are independent sets of X .*

Proof. Using Lemma 23 and Lemma 4 we wrote a Sage program generating all possible graphs on $\{k_1, \dots, k_5\}$ and 8 triangles, each contained in a different set $X_{i,j}$, for $1 \leq i < j \leq 5$. There are 2 non isomorphic ways to chose 8 sets for triangles (among 10 sets). All the obtained graph in one configuration do not interlace X , while in the other configuration only one non-isomorphic examples were found giving rise to 117 comparability graphs. None of them has a clique of order 56. Thus the lemma follows. \square

We are now able to prove our main theorem. The lists of graphs obtained in this part are too large to be hosted online hence they are not included in Table 1. However they can be obtained by a request to the authors.

Theorem 1. *The graph X does not exist.*

Proof. By the previous lemma, at least 3 graphs among $X_{i,j}$, for $1 \leq i < j \leq 5$, are independent sets. It is an easy check that there are 4 non-isomorphic configurations for the choice of 3 sets among $X_{i,j}$. These are: $(X_{1,2}, X_{2,3}, X_{4,5}), (X_{1,2}, X_{2,3}, X_{1,3}), (X_{1,2}, X_{2,3}, X_{2,4}), (X_{1,2}, X_{2,3}, X_{3,4})$. Notice that in all combinations we have sets $X_{1,2}, X_{2,3}$.

First we analyze the possible candidates for graphs induced by $\{k_1, \dots, k_5\} \cup X_{1,2} \cup X_{2,3}$. We do this by generating all bipartite graphs that can represent $X_{1,2} \cup X_{2,3}$.

This is done using McKay's program *genbg*. Adding the vertices $\{k_1, \dots, k_5\}$ and removing non-interlacing graphs we end up with a list of 654325 graphs. By computing $sc(G)$ for every such graph G and extending it to have order 19 we end up with a list of 361547477 star complements. By computing their respective comparability graphs and removing isomorphisms we end up with about 100^6 graphs. We have verified that none of these graphs has a clique of order 56 hence implying our assertion. \square

5 Computational aspects

In this section we briefly describe the computational tools and resources used to produce our result. As described in Section 2.3, our approach required generating a list of candidates for an induced subgraph of X , compute their comparability graphs and check their clique numbers. Most programs were written and tested independently in C and Sage, however most of the computation was performed only by C programs due to their efficiency.

5.1 Extending graphs and computing comparability graphs

As described in Section 2 we generated a list of graphs \mathcal{L} such that if X exists then one of the graphs in \mathcal{L} must be an induced subgraph of X . In order to rule out the existence of X we had to obtain star complements for each of the graphs in \mathcal{L} and check the clique number of its comparability graphs. Some of the graphs in \mathcal{L} already had star complements as induced subgraphs and were easy to handle. However some of the graphs did not, and in this case we had to find maximal induced subgraphs with no 2 as eigenvalue, and extend them to have order 19. When choosing these subgraphs we tried to maximize the order of the automorphism group while minimizing the number of obtained subgraphs - note that two non isomorphic members of \mathcal{L} may have isomorphic subgraphs. Both lists \mathcal{L} and the one obtained from it are available on the GitHub page.

Let us remark that we have found out that the process of extending graphs is computationally feasible whenever the obtained subgraphs have order 17. For otherwise we obtained far too many graphs of order 19.

The task of extending graphs to have order 19 was done by the already introduced program *extend.c* which takes as input a file with graphs given in graph6 string format and for each graph outputs all possible ways to introduce a new vertex to it so that the newly obtained graph interlaces X . If the input graph has a graceful pair of vertices then the extensions giving the minimal amount of graphs is written. Again, graphs are written in graph6 format.

After each iteration of *extend.c* we used McKay's *shortg* [15] program to remove isomorphic graphs from the obtained lists. Extending a graph of order 18 takes roughly 0.5 seconds on a standard desktop machine and the whole computation for the proof of our main took roughly 240 CPU hours.

To compute comparability graphs we wrote a graph that takes as input a list of star complements and for each output writes the graph6 representation of its comparability graph. The program is called *compGraph2graph6.c* and

is found on GitHub [1]. We have found that the average comparability graph gets computed in 0.5 seconds and hence the instances of Theorem 1 were computed in about 5000 CPU hours. Let us note that the program does not output comparability graphs with order smaller than the clique number sought - in our case 57.

The computationally most intensive part was computing the clique number of the obtained comparability graph. This step took roughly 150000 CPU hours and was carried on a computational grid of 2000 CPU's.

Both programs use the GNU GSL library for linear algebra routines and make use of the precision guaranteed by their implementation. Finally let us remark that in some of the steps we made use of the GNU parallel program [21].

5.2 Computing the clique number

While it is in general hard to compute the clique number of a graph, the structure of comparability graphs makes this task a little easier. As one may suspect by its definition, Comp offers a lot of symmetry which we exploit as follows. In what follows, $G[N(v)]$ will denote the subgraph of G induced by the neighbors of a vertex $v \in V(G)$.

Suppose we wish to compute the clique number $\omega(G)$ and let $v \in V(G)$. Then either v is contained in a maximal clique of G or is not. In the latter case the maximal clique of G equals the maximal clique in $\text{Comp} - v$. In other words

$$\omega(G) = \max(\omega(G) - v, \omega(G[N(v)])) + 1).$$

Now, the key fact in computing $\omega(\text{Comp})$ is that its automorphism group is fairly large and hence in computing its clique number we can remove the entire orbit $o(v)$ of a vertex v . Given that $o(v)$ is fairly large, the obtained graph $\text{Comp} - o(v)$ is much smaller. We need not stop here. The key property that is used in the above idea is the fact that if $u, v \in V(G)$ are in the same orbit of $\text{Aut}(G)$, then the graphs $G[N(u)]$ and $G[N(v)]$ are isomorphic. For our purposes we can define the *extended orbit of a graph G* as a partition of $\tilde{O}(G)$ of $V(G)$ such that two vertices u, v are in the same part if and only if $G[N(u)] \cong G[N(v)]$. Summarizing the above idea into pseudo code we designed the following algorithm.

In order to compute the clique number of our comparability graphs we used a variant of Algorithm 1 which leaves out two major details. Namely the computation of the extended orbits of G and the *cliqueNumberBruteforce* routine. For the later, we needed an established program that calculates the

Algorithm 1 Algorithm for computing clique numbers of symmetric graphs

```
1: procedure CLIQUENUMBER( $G, c$ )
2:    $cl \leftarrow 0$ 
3:   while  $|V(G)| > c$  do
4:      $\tilde{\mathcal{O}} \leftarrow \text{extendedOrbits}(G)$ 
5:     if  $|\tilde{\mathcal{O}}| = |V(G)|$  then
6:       break
7:      $o \leftarrow$  some orbit of  $\tilde{\mathcal{O}}$ 
8:      $v \leftarrow$  an element of  $o$ 
9:      $cltmp \leftarrow \text{cliqueNumber}(G[v]), c) + 1$ 
10:    if  $cltmp > cl$  then
11:       $cl \leftarrow cltmp$ 
12:     $G \leftarrow G - o$ 
  return  $\max(cl, \text{cliqueNumberBruteForce}(G))$ 
```

clique number of a graph. We have found out that on our instances the program *mcqd* [13] drastically outperforms the well known clique finding algorithm *Cliquer* [18]. Hence whenever our input graph is small enough, we simply use *mcqd*. Since we only need to determine whether our graph has a clique of size at least 57 or not we made use of an additional optimization. Suppose we are trying to decide whether a graph G has a clique of size k and the greedy coloring algorithm shows that we can properly color the vertices of G using less than k colors. Then the clique number of G is smaller than k and we can stop our search. This is the essential idea behind the implementation of *mcqd* and we used it to obtain an even more efficient test for comparability graphs.

The second problem of computing the extended orbits is reduced to the problem of computing the orbits of the automorphism groups and canonical forms of graphs. While the computational complexity of these two problems is not settled, it is well known that in practice both problem offer efficient practical solutions. For example, it takes Bliss [12] about 5 seconds of CPU time on a standard laptop to compute the full automorphism group of a typical comparability graph of order 6000 and density 0.4.

In order to compute the extended orbits of a graph G we first compute the orbits \mathcal{O} of its automorphism group. Finally for every representative of \mathcal{O} we compute the canonical form of $G[N(v)]$ and join orbits with equal canonical forms.

A simple implementation of the above algorithm was implemented in Sage and is available on the GitHub repository under the name *cliqueNum-*

ber.sage. We used a more efficient *C++* implementation that we will present in a subsequent paper.

Finally, let us remark that both *mcqd* and *Bliss* were integrated into Sage for the purposes of this paper.

6 Final remarks

We have shown that a $(75, 32, 10, 16)$ SRG does not exist by presenting a classification approach based on the star complement technique. The main property that we exploited was the fact that such a SRG has an eigenvalue of high multiplicity, namely 56 which implies that the star complement graph is small (19 vertices). Thus one can avoid the combinatorial explosion of constructing all possible star complements, provided that one can build large enough induced structure for the star complement graph. In our case this was established by building the star complement around a maximal clique of our SRG. Two things were crucial for our approach to work. First was the fact that many of the obtained comparability graphs were isomorphic thus significantly reducing the number of graphs whose clique number was to be determined. The second crucial part was the fact that comparability graphs had large automorphism groups thus allowing to exploit their symmetries when computing their clique number. We believe that a similar approach can be used to classify at least one of the following open parameters $(69, 20, 7, 5)$, $(95, 40, 12, 20)$, $(96, 45, 24, 18)$, $(99, 42, 21, 15)$.

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